Column Analogy Method

COLUMN ANALOGY METHOD

Background

Knowledge is needed from the following:

- Statics: Areas, Centroids, Moment of Inertia, etc.
- Strength of Materials: Stresses, Stress formula, etc..
- Structural Analysis I: Determinate Structures, etc..

Objectives

The method is used to calculate:

- Stiffnesses, Carry over factors, Fixed End moments
- Simple frame problems and beam problems
COLUMN ANALOGY METHOD

1 GENERAL

Column analogy was the second outstanding contribution to the structural field made by the late Hardy Cross. The method is applicable to the analysis of structures statically indeterminate to not more than the third degree, including fixed-ended beams, single-span arches and frames, and closed boxes. Another application of column analogy, and perhaps the most useful today, is the calculation of the carry-over factors, stiffness factors, and fixed-end moments necessary for analyzing structures with members of varying moments of inertia by moment distribution.

1 GENERAL

The conjugate-beam procedure is an analogy in that a convenient viewpoint was taken of the mechanical operations involved in figuring the deflections in members subject to bending moment. The $M/EI$ diagram was placed on a fictitious structure in which the resulting shear and moments coincided exactly with the slopes and deflections in the original structure.

In a similar manner the column-analogy procedure takes a convenient view of the operations involved in figuring the moments in a statically indeterminate structure. The analogy pertains to the identities existing between the moments produced in a statically indeterminate structure and the stresses produced in an eccentrically loaded short column. The computations are reduced to a nearly mechanical procedure, and the comment, "I just put the figures in a table, push the button of the calculator, and out comes the answer," is sometimes heard.
2 DEVELOPMENT OF THE METHOD

For the following discussion a short prismatic column, for which the deflections caused by bending are negligible, is considered. The column is shown in Fig. 1 loaded with an eccentric load $P$, and a stress diagram showing the variation of stress across the cross section of the column is drawn. The load is assumed to be applied at a point along the $X$ axis of the cross section a distance $e$ from the $Y$ axis. The following expression may be written for the stress at any point a perpendicular distance $y$ from the $Y$ axis:

$$ f = \frac{P}{A} \mp \frac{Mc}{I_y} = \frac{P}{A} \mp \frac{Pe_y}{I_y} $$

An important fact to notice is that the load $P$ coincides with the centroid of the forces produced in the column cross section or with the centroid of the stress diagram.

The fixed-ended beam of Fig. 2, which supports a concentrated load $P$, is now considered, and the magnitude of the end moments is desired. The beam is assumed to be replaced with a short column having a cross section with a centerline the same in shape and length as the beam. Its width at any section equals the $1/EI$ value of the beam at the corresponding section.
N is the resultant of $M/EI$ area

\[ f = N \pm \frac{Mc}{I_y} = N \pm \frac{Ney}{I_y} \]

**Figure 1**

**Example 1**

\[
N = \left(\frac{1}{2}\right)\frac{200}{EI} \cdot \frac{3000}{EI} = \frac{30000}{EI}
\]

\[
A = \frac{l}{EI} = \frac{30}{EI}
\]

\[
\epsilon = \frac{l}{2} - \frac{l + a}{3} = 15 - \frac{30 + 10}{3} = 1.67
\]

\[
i = \frac{l^3}{12EI} - \frac{30^3}{12EI} = \frac{2250}{EI}
\]

\[
y = \frac{l}{2} = 15
\]

\[
M_A = \frac{3000/EI}{30/EI} + \frac{(3000/EI)(1.67)(15)}{2250/EI} = 133.3^4
\]

\[
M_B = \frac{3000/EI}{30/EI} - \frac{(3000/EI)(1.67)(15)}{2250/EI} = 66.7^4
\]
Example 2
Compute the Fixed End Moments

Solution

\[ N = \left(\frac{1}{2}\right)(6) \left(\frac{0.30}{E}\right)(2) + (6) \left(\frac{0.60}{E}\right) + \left(\frac{1}{3}\right)(6) \left(\frac{0.30}{E}\right) = \frac{6.30}{E} \]

\[ A = (6) \left(\frac{0.005}{E}\right)(2) + (6) \left(\frac{0.01}{E}\right) - \frac{0.12}{E} \]

\[ e = 0 \]

\[ M_x = M_y = \frac{N}{A} = \frac{Ney}{I} = \frac{6.30/E}{0.12/E} = 52.5 \]

Example 3
Compute the Carry over factor from A to B.

\[ N = 1.0 \]

\[ A = \frac{30}{EI} \]

\[ I = \frac{l^3}{12EI} = \frac{2250}{EI} \]

\[ e = \frac{l}{2} = 15 \]

\[ y = \frac{l}{2} = 15 \]
Example 3

Compute the Stiffness at A and the carry over factor from A to B.

Solution

\[ M_A = \frac{N}{A} + \frac{Ney}{I} = \frac{1.0}{30/EI} + \frac{(1.0)(15)(15)}{2250/EI} = +0.1333EI \]

\[ M_B = \frac{N}{A} - \frac{Ney}{I} = \frac{1.0}{30/EI} - \frac{(1.0)(15)(15)}{2250/EI} = -0.0667EI \]

\[ C_{AB} = \frac{M_B}{M_A} = -\frac{0.0667EI}{0.1333EI} = -\frac{1}{2} \]
Example 5
Compute the Fixed End Moments, Carry over factors and the stiffnesses.

Solution
### Example 5

<table>
<thead>
<tr>
<th>SECTION</th>
<th>$N$</th>
<th>$A$</th>
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<tbody>
<tr>
<td>1</td>
<td>$(10)(2)$</td>
<td>$3.24$</td>
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<tr>
<td></td>
<td>$6.162EI_0$</td>
<td>$EI_0$</td>
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<tr>
<td>2</td>
<td>$(30)(2)$</td>
<td>$17.80$</td>
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<tr>
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<tr>
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<td>$(90)(2)$</td>
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<td>$\Sigma$</td>
<td>$404.04$</td>
<td>$6.174$</td>
</tr>
<tr>
<td></td>
<td>$EI_0$</td>
<td>$EI_0$</td>
</tr>
</tbody>
</table>

### Example 5

\[
N = 2 \times \frac{404.04}{EI_0} = \frac{808.08}{EI_0}
\]
\[
A = 2 \times \frac{6.174}{EI_0} = \frac{12.348}{EI_0}
\]
\[
I = \left(\frac{1}{12}\right)\left(\frac{0.162}{EI_0}\right)(20)^3 + \left(\frac{1}{12}\right)\left(\frac{0.134}{EI_0}\right)(16)^3 + \left(\frac{1}{12}\right)\left(\frac{0.333}{EI_0}\right)(12)^3
\]
\[
+ \left(\frac{1}{12}\right)\left(\frac{0.371}{EI_0}\right)(8)^3 = \frac{217.3}{EI_0}
\]
Example 5

Fixed-end moments:

\[ e = 0 \]
\[ M_A - M_B = \frac{808.08/EI_0}{12.348/EI_0} = 65.4^k \]

Carry-overs and stiffness factors:

\[ N = 1.0 \]
\[ e = 10.0 \]
\[ M_A = \frac{1.0}{12.348/EI_0} + \frac{(1.0)(10.0)(10.0)}{217.3/EI_0} = +0.081EI_0 + 0.46EI_0 = +0.541EI_0 \]
\[ M_B = +0.081EI_0 - 0.46EI_0 = -0.379EI_0 \]
\[ C_{AB} = C_{BA} = \frac{-0.379EI_0}{+0.541EI_0} = -0.701 \]
\[ K_A = K_B = 0.541EI_0 \]

Example 6

Distribute the Fixed End Moments for the beam for which the stiffnesses and carry over factors and fixed end moments have been calculated.
Example 6

Solution

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<tr>
<th></th>
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<th>K</th>
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<tr>
<td>+0.75</td>
<td>0.78</td>
<td>-0.78</td>
<td>0.75-</td>
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<td>17.1</td>
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<td>22.8</td>
<td>17.1</td>
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FEM

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<th>0.57</th>
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<tr>
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Analysis of Frames

![Analysis of Frames Diagrams](image)

*Images and diagrams not directly transcribed due to the complexity of the visual content.*
Example 7
Determine the moments at C and D.
Remove the fixed end at A.
Solution

\[ M_C = \frac{2000}{46.67} + \frac{(2000)(10)(10)}{4222} - \frac{(2000)(8.09)(8.58)}{1904} + M_{\text{single beam}} \]
\[ M_C = +42.8 + 47.4 - 72.8 + 0 = 17.4^k \]

Moment at C:

\[ M_D = \frac{2000}{46.67} + \frac{(2000)(10)(10)}{4222} - \frac{(2000)(8.09)(11.42)}{1904} - 400 \]
\[ M_D = +42.8 + 47.4 + 96.9 - 400 = -212.9^k \]